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Effects of an Oscillating Field on a Diffusion Process in the Presence of a Trap

Moshe Gitterman,[1,2] Shlomo Havlin,[1,2] and George H. Weiss[1]

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Consider a diffusion process on an infinite line terminated by a trap and modulated by a periodic field. When the frequency is equal to zero the mean time to trapping will be finite or infinite, depending on the sign of the field. We ask whether this behavior can be changed by an oscillatory field, and show that it cannot for pure Brownian motion. We suggest that transition can appear when the signal propagation velocity is finite as for the telegrapher's equation. We further suggest that the asymptotic time dependence of the survival probability is proportional to $t^{-1/2}$ just as in the case of ordinary diffusion. The same conclusion is shown to hold for a system whose dynamics is governed by the equation $\dot{x} = xv(t)/L$, where L is a constant.

KEY WORDS: Brownian motion; stochastic resonance; trapping processes.

1. INTRODUCTION

The phenomenon of stochastic resonance was originally proposed to account for the periodicity of the earth's ice ages,[1-4] and has only recently been investigated with some intensity in the literature of statistical physics (cf., for example, refs. 5 and 6). In this phenomenon the periodic excitation of a parameter characterizing a dynamical system can lead to sometimes dramatic changes in the behavior of the system as a function of the driving frequency. Thus, for example, it has been shown that in a noisy nonlinear bistable system one can actually increase the signal-to-noise ratio (defined in terms of the power output of the system) by periodically modulating the depth of the wells at a resonant frequency.[6] Stochastic resonance has also been observed in experimental systems.[7,8]

We have recently examined a very simple random walk model which exhibits a form of stochastic resonance in terms of the mean time to

[1] National Institutes of Health, Bethesda, Maryland 20892.
[2] Permanent address: Department of Physics, Bar-Ilan University, Ramat-Gan, Israel 52100.

trapping.[9] In this model one considers random walkers moving along a line (or particles diffusing on a line) terminated at both ends by a trap. The transition probabilities (or the convective velocity in the case of diffusion) are varied periodically and one asks how the mean first passage time (MFPT) to trapping behaves as a function of the frequency. Our results indicated that there is a "resonant" frequency which minimizes this time. Reichl has also observed what might be interpreted as a phase transition in the eigenvalues characterizing a one-dimensional random walk system in the presence of reflecting boundaries.[10]

In this note we discuss the problem of whether, and under what circumstances, a periodic excitation can lead to a phase transition in the mean trapping time for a diffusion process on an infinite line with a trap at $x=0$. In order to specify the sense in which we use the term "phase transition," let us suppose that the dynamics of the system is described by the equation

$$\dot{x} = V\sin(\omega t + \phi) + w(t) \tag{1}$$

where V is a constant amplitude, ϕ is a constant phase, and $w(t)$ is white noise. One knows that when $\omega = 0$ the MFPT to trapping will be infinite provided that $V\sin\phi \geqslant 0$ and finite when $V\sin\phi < 0$. Thus, in this trivial case there is an identifiable transition in the behavior of the MFPT. The question motivating the present investigation is whether one can find a phase transition of this sort at some other value of ω that differs from 0. We will show that the answer to this question is negative for a diffusion process although it may be possible for a process with a finite signal transmission speed. A second question is whether one can find such a phase transition by making an appropriate change in the dynamics of the system. We have been able to find an explicit solution to the resulting problem for the specific case in which the velocity in the noise-free system is proportional to x, but in no other case. Our results lead us to the conjecture that additive white noise cannot cause a phase transition if there is none in the noise-free system.

2. BROWNIAN MOTION ON A LINE

If Eq. (1) describes the system behavior, one can describe the dynamical behavior of the system in terms of a probability density $p(x, t \mid x_0, 0)$ for the position of the diffusing particle at time t, given an initial position x_0. This is the solution to the diffusion equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - v(t)\frac{\partial p}{\partial x} \tag{2}$$

where, for our present problem, $v(t) = V\sin(\omega t + \phi)$, D is a diffusion constant, and the probability density for the displacement is to be found subject to the initial condition $p(x, 0 \mid x_0, 0) = \delta(x - x_0)$. Because of the presence of a trap at $x = 0$ the solution to this equation must satisfy the boundary condition $p(0, t \mid x_0, 0) = 0$. However, one cannot use the method of images or any modification thereof to find a solution as would be the case if $v(t)$ were a constant. It is possible to find an alternate formulation of the problem by introducing a new coordinate ξ by

$$\xi = x - \int_0^t v(\tau)\, d\tau \tag{3}$$

in which case Eq. (2) is transformed into the equivalent form of an ordinary diffusion equation

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial \xi^2} \tag{4}$$

Although the transformation in Eq. (3) appears to lead to a simplification of the problem, in reality it merely transfers the difficulty in solving the problem to the fact that the boundary condition now depends on time. In the new set of coordinates, the transformation in Eq. (3) changes the boundary condition to a time-dependent one:

$$p\left(-\int_0^t v(\tau)\, d\tau, t \mid x_0, 0 \right) = 0 \tag{5}$$

While there is a considerable mathematical literature on Brownian motion in the presence of time-dependent boundaries,[11-16] we need not make use of it to determine whether the MFPT till trapping is finite or infinite, but rather we rely on a simple argument based on the fact that in Brownian motion to a fixed trapping point the MFPT is infinite.

The argument consists of two parts. In the first we will suppose that $V < \omega x_0$. In the absence of noise the limits of $x(t)$ are $x_0 \pm V/\omega$, which implies that when $V < \omega x_0$ a particle will never be trapped and in the contrary case it will always be trapped. In our analysis we can picture the motion of the boundary in the (x, t) plane as shown in Fig. 1. Tangent lines have been drawn to the maximum and minimum displacements of the boundary along the x coordinate, i.e., $x_{\max} = V/\omega$ and $x_{\min} = -V/\omega$, both of which are less than x_0. We now argue that since the mean time for a particle at x_0 to reach either tangent line is infinite, it is *a fortiori* true that the time to reach the oscillating boundary is infinite. The second part of the argument accounts for the possibility $V \geq \omega x_0$. Here we note that since the effective speed of propagation of a signal for a diffusion process is infinite,[17,18] there is a nonzero probability that the particle initially at x_0 will reach some point outside of the tangent line before reaching the trap. Such a particle will take an infinite amount of time, on average, to be trapped. Therefore, the mean time for an arbitrary particle to be trapped

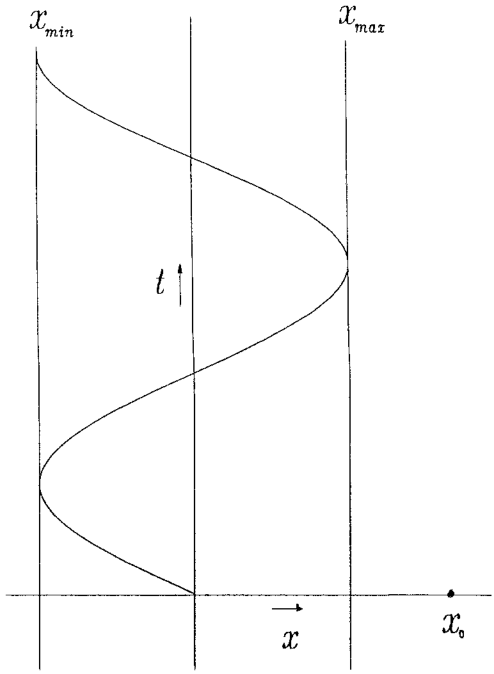

Fig. 1. Schematic representation of a diffusion system with an oscillating boundary trapping point.

must likewise be infinite. Thus we see that the effect of additive noise on the dynamical system is to eliminate the phase transition that appears in the deterministic formulation. We note that the mean trapping time is necessarily infinite provided that there exists a value of $x_{\max}$ that is strictly less than x_0. The corresponding field is not restricted to be a periodic one provided that the condition stated above is fulfilled.

Our argument has so far been directed toward determining whether the MFPT to trapping is finite or infinite. A related question is that of determining the asymptotic decay of the probability that the particle is untrapped at time t. A simple physical argument suggests that this decay should go like $t^{-1/2}$ at sufficiently long times. The argument starts from the observation that if the only trapping boundaries in the problem are the dotted lines in Fig. 1, then, in the case of trapping at either boundary the survival probability will go asymptotically to zero like $t^{-1/2}$, although the constant multipliers will differ. This suggests that the survival probability $S(t \mid x_0)$ satisfies the inequality

$$a \leqslant t^{1/2} S(t \mid x_0) \leqslant b \tag{6}$$

at sufficiently long times, where a and b are two constants appropriate to trapping at the two boundaries. An anonymous referee has kindly provided a more complete, rigorous proof of the asymptotic $t^{-1/2}$ behavior for the diffusion process.

The infinite MFPT at all values of ω is clearly due to the infinite velocity of signal transmission associated with Brownian motion. One expects that if the underlying process is characterized by a finite speed of transmission, as exemplified by solutions of the telegraph equation,[17,18] it might lead to a minimum in the MFPT akin to that reported in reference 9.

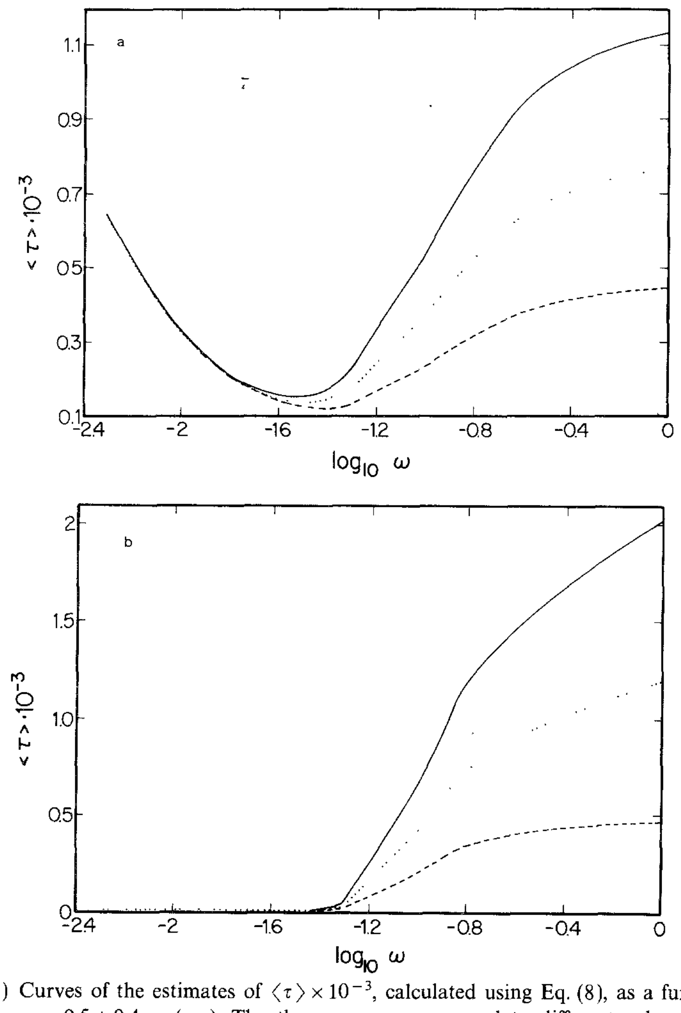

Fig. 2. (a) Curves of the estimates of $\langle\tau\rangle \times 10^{-3}$, calculated using Eq. (8), as a function of $\log_{10}\omega$ for $p_{\pm} = 0.5 + 0.4\cos(\omega n)$. The three curves correspond to different values of N in Eq. (8): (--) $N = 4000$; ($\cdots$), $N = 10{,}000$; (—) $N = 20{,}000$. (b) Curves of the estimates of $\langle\tau\rangle \times 10^{-3}$ as a function of $\log_{10}\omega$ for $p_{\pm} = 0.5 - 0.4\cos(\omega n)$, with the same labeling convention as in part (a).

To test this hypothesis we have used the method of exact enumeration to simulate the behavior of the MFPT as a funtion of frequency for a lattice random walk in which the transition probabilities that take the random walk to one of the two nearest neighbors are chosen as

$$p_{\pm}(n)=\tfrac{1}{2}\pm\varepsilon\cos(\omega n) \tag{7}$$

where n is the step number. Since this simulation is necessarily carried out in discrete time and only allows for finite step sizes, the system is one with a finite velocity of signal propagation. In Fig. 2 we present some results for the MFPT, $\langle\tau\rangle$, plotted as a function of the frequency ω. The parameter $\langle\tau\rangle$ was estimated by calculating the survival probability, $S(n)$, as a function of n, and performing the summation

$$\langle\tau\rangle\sim\sum_{n=1}^{N} S(n) \tag{8}$$

Two cases are shown: Fig. 2a presents the result for $\langle\tau(\omega)\rangle$ for the parameter value $\varepsilon=0.4$ and Fig. 2b gives the results for $\varepsilon=-0.4$. Because of the cosine term in Eq. (7) the first case represents random walkers that initially move away from the origin and the second to random walkers that initially move towards the origin. The different curves in each figure correspond to different values of N in Eq. (8). In Fig. 2a we see that there is indeed a minimum present in the curves of the estimates of MFPT as a function of ω similar to that found in reference 9. We interpret this as an indication of a genuine minimum that can occur in a system with a finite propagation speed for signals. We also see a separation, at higher ω, between the curves for different N in both parts of Fig. 2. This clearly indicates the divergence found in the case of Brownian motion. In Fig. 2b, for $\varepsilon=-0.4$, the very low frequency behavior of $\langle\tau\rangle$ tends towards a constant, while at higher frequencies the estimates of $\langle\tau\rangle$ tend to diverge. These simulations tend to substantiate our understanding that the key element (in the absence of an overall bias) in determining the behavior of $\langle\tau\rangle$ as a function of ω is the velocity of signal propagation.

3. MOTION IN A BIASING FIELD

Our results so far pertain only to Brownian motion in the presence of a field that is time-dependent but not space-dependent. A related question concerns possible effects of noise on systems in the presence of a combined time- and space-dependent field of force. We will specifically consider the effect of noise applied to a deterministic system in which a particle, initially at x_0, never reaches the origin. This is equivalent to saying that the time for the moving particle to reach the origin is always infinite. Is it possible for the corresponding MFPT to become finite in the presence of a timevarying field? We have not answered this question in generality, but can address the problem for a diffusing particle whose displacement $x(t)$ satisfies the stochastic differential equation

$$\dot{x} = \frac{x}{L}\, v(t) + w(t) \tag{9}$$

where L is a constant with the dimensions of length and $v(t)$ is a time-dependent velocity. In the absence of noise the solution to this equation is

$$x(t) = x_0 \exp\left[\frac{1}{L}\int_0^t v(\tau)\, d\tau\right] \tag{10}$$

so that $x(t) \neq 0$ for finite t provided that the integral $\int_0^t v(\tau)\, d\tau > -\infty$.

To solve for the survival probability in the presence of noise we let $p(x, t \mid x_0, 0)$ be the probability density for the random variable $x(t)$ which satisfies $x(0) = x_0$. The Smoluchowski equation corresponding to Eq. (9) is the following:

$$\frac{\partial p}{\partial t} = D\, \frac{\partial^2 p}{\partial x^2} - \frac{v(t)}{L}\, \frac{\partial}{\partial x}(xp) \tag{11}$$

which must satisfy the boundary condition $p(0, t \mid x_0, 0) = 0$ as well as the initial condition $p(x, 0 \mid x_0, 0) = \delta(x - x_0)$. The probability that the particle has not reached the origin by time t, $S(t \mid x_0)$, is related to $p(x, t \mid x_0, 0)$ by

$$S(t \mid x_0) = \int_0^\infty p(x, t \mid x_0, 0)\, dx \tag{12}$$

Equation (11) is readily solved using a Fourier sine transform and $S(t \mid x_0)$ can be expressed in terms of the normal integral

$$\Phi(x) \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2}\, du \tag{13}$$

as

$$S(t \mid x_0) = 2\Phi\left(\frac{x_0}{[2Dh(t)]^{1/2}}\right) - 1 \tag{14}$$

which means that the form of the survival probability remains unchanged from that obtained with simple diffusion, but the time is rescaled. In this last equation the function $h(t)$ is defined in terms of a dimensionless displacement

$$\chi(t) = \frac{1}{L}\int_0^t v(\tau)\, d\tau \tag{15}$$

as

$$h(t) = \int_0^t \exp[2\chi(\tau)]\, d\tau \tag{16}$$

Since the MFPT to the origin $\langle\tau\rangle$ is related to $S(t\,|\,x_0)$ by

$$\langle\tau\rangle = \int_0^\infty S(t\,|\,x_0)\,dt \tag{17}$$

we see that the convergence of the integral depends on how $S(t\,|\,x_0)$ goes to 0 in the limit of large time. When $v(t)$ is oscillatory the function $\chi(t)/t$ remains bounded in the limit $t \to \infty$, which implies the result that $S(t\,|\,x_0)$ goes asymptotically to zero as $t^{-1/2}$, analogous to the case described earlier in which no field is present.

In summary, our results lead us to conjecture that the oscillatory field will have no influence on whether the mean time to be trapped at the origin is finite or infinite. This is intuitively clear in the case of a field that varies with a sufficiently high frequency. However, it may be the case that when the time to reach the trapping point is finite in the noise-free case, as for example, when diffusion is described by a telegrapher's equation,[18] there will always be a type of stochastic resonance that exists as a function of frequency. We have not investigated this point except in the two-trap case[9] cited earlier.

We are indebted to a referee for providing a rigorous proof of the asymptotic proportionality of the survival time to $t^{-1/2}$ in the case of ordinary diffusion.

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